WeierstrassSigma4

Notations

Traditional name

Associated Weierstrass sigma function

Traditional notation

$\sigma_n(z, g_2, g_3)$

Mathematica StandardForm notation

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WeierstrassSigma[n, z, {g2, g3}]
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Primary definition

09.16.02.0001.01

$$\sigma_n(z, g_2, g_3) \equiv \frac{e^{-\eta_n z}\, \sigma(z+\omega_n; g_2, g_3)}{\sigma(\omega_n; g_2, g_3)} \,/;$$

$$n \in \{1, 2, 3\} \land \{\omega_1, \omega_2, \omega_3\} \equiv \{\omega_1(g_2, g_3), -\omega_1(g_2, g_3) - \omega_3(g_2, g_3), \omega_3(g_2, g_3)\} \land \eta_n \equiv \zeta(\omega_n; g_2, g_3) \land n \in \{1, 2, 3\}$$

Special notations for this file:

09.16.02.0002.01

$$\{\omega_1, \omega_3\} \equiv \{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\}$$

09.16.02.0003.01

$$\{\omega_1, \omega_2, \omega_3\} \equiv \{\omega_1(g_2, g_3), -\omega_1(g_2, g_3) - \omega_3(g_2, g_3), \omega_3(g_2, g_3)\}$$

09.16.02.0004.01

$$e_n \equiv \wp(\omega_n; g_2, g_3) \land n \in \{1, 2, 3\}$$

09.16.02.0005.01

$$\eta_n \equiv \zeta(\omega_n; g_2, g_3) \land n \in \{1, 2, 3\}$$

09.16.02.0006.01

$$q \equiv \exp\left(\frac{\pi\, i\, \omega_3}{\omega_1}\right)$$

Specific values

Specialized values

For fixed *z*

Degenerate case:

09.16.03.0001.01

$$\sigma_n(z; 0, 0) == 1 \;/;\; n \in \{1, 2, 3\}$$

For fixed $\{g_2, g_3\}$

Values at half-periods

09.16.03.0002.01

$$\sigma_j(0; g_2, g_3) == 1 \;/;\; j \in \{1, 2, 3\}$$

09.16.03.0003.01

$$\sigma_j(2 m \omega_1 + 2 n \omega_3; g_2, g_3) == \tilde{\infty} \;/;\; \{m, n\} \in \mathbb{Z} \land j \in \{1, 2, 3\}$$

09.16.03.0004.01

$$\sigma_n(\omega_j; g_2, g_3) == 0 \;/;\; j \in \{1, 2, 3\}$$

09.16.03.0005.01

$$\sigma_i((2 m + 1) \omega_i + 2 n \omega_j + 2 r \omega_k; g_2, g_3) == 0 \;/;\; \{m, n, r\} \in \mathbb{Z} \land \{i, j, k\} \in \{1, 2, 3\} \land i \neq j \neq k$$

09.16.03.0006.01

$$\sigma_i(\omega_j; g_2, g_3) == e^{-\eta_i \omega_j} \frac{\sigma(\omega_k; g_2, g_3)}{\sigma(\omega_i; g_2, g_3)} \;/;\; \{i, j, k\} \in \{1, 2, 3\} \land i \neq j \neq k$$

09.16.03.0007.01

$$\sigma_j(\omega_i; g_2, g_3)\, \sigma_k(\omega_i; g_2, g_3) == e^{\eta_i \omega_i} \;/;\; \{i, j, k\} \in \{1, 2, 3\} \land i \neq j \neq k$$

09.16.03.0008.01

$$\frac{\sigma_i(\omega_k; g_2, g_3)}{\sigma_i(\omega_j; g_2, g_3)\, \sigma_k(\omega_i; g_2, g_3)} == e^{\eta_j \omega_i} \;/;\; \{i, j, k\} \in \{1, 2, 3\} \land i \neq j \neq k$$

09.16.03.0009.01

$$\frac{\sigma(\omega_k; g_2, g_3)}{\sigma(\omega_j; g_2, g_3)\, \sigma_j(\omega_i; g_2, g_3)} == e^{\eta_j \omega_i} \;/;\; \{i, j, k\} \in \{1, 2, 3\} \land i \neq j \neq k$$

09.16.03.0010.01

$$\frac{\sigma_j(\omega_i; g_2, g_3)^2}{\sigma(\omega_i; g_2, g_3)^2} == e_i - e_j \;/;\; \{i, j\} \in \{1, 2, 3\} \land i \neq j$$

General characteristics

Domain and analyticity

$\sigma_n(z, g_2, g_3), n \in \{1, 2, 3\}$ are an entire analytical functions of z, g_2, and g_3, which are defined in $\mathbb{C}^3$.

09.16.04.0001.01

$$(n * z * \{g_2 * g_3\}) \longrightarrow \sigma_n(z, g_2, g_3) :: (\{1, 2, 3\} \otimes \mathbb{C} \otimes \{\mathbb{C} \otimes \mathbb{C}\}) \longrightarrow \mathbb{C}$$

Symmetries and periodicities

Parity

$\sigma_n(z, g_2, g_3)$, $n \in \{1, 2, 3\}$ are an even functions with respect to z.

09.16.04.0002.01

$$\sigma_n(-z; g_2, g_3) = \sigma_n(z; g_2, g_3) \;/;\; n \in \{1, 2, 3\}$$

Mirror symmetry

09.16.04.0003.01

$$\sigma_n(\bar{z}; \overline{g_2}, \overline{g_3}) = \overline{\sigma_n(z; g_2, g_3)} \;/;\; n \in \{1, 2, 3\}$$

Periodicity

$\sigma_n(z, g_2, g_3)$, $n \in \{1, 2, 3\}$ are a quasi-periodic functions with respect to z.

09.16.04.0004.01

$$\sigma_n(z + 2\omega_n; g_2, g_3) = -e^{2\eta_n (z+\omega_n)} \sigma_n(z; g_2, g_3) \;/;\; n \in \{1, 2, 3\}$$

09.16.04.0005.01

$$\sigma_n(z + 2\omega_j; g_2, g_3) = e^{2\eta_j (z+\omega_j)} \sigma_n(z; g_2, g_3) \;/;\; \{n, j\} \in \{1, 2, 3\} \land n \neq j$$

09.16.04.0006.01

$$\sigma_i(z + 2 m \omega_i + 2 n \omega_j + 2 r \omega_k; g_2, g_3) = (-1)^{n r + r m + m n + m} e^{2 (m \eta_i + n \eta_j + r \eta_k)(z + m \omega_i + n \omega_j + r \omega_k)} \sigma_i(z; g_2, g_3) \;/;$$
$$\{i, j, k\} \in \{1, 2, 3\} \land i \neq j \neq k \land \{m, n, r\} \in \mathbb{Z}$$

Transformation of half-periods

09.16.04.0007.01

$$\sigma_1(z; g_2(a\omega_1 + b\omega_3, c\omega_1 + d\omega_3), g_3(a\omega_1 + b\omega_3, c\omega_1 + d\omega_3))$$
$$\sigma_2(z; g_2(a\omega_1 + b\omega_3, c\omega_1 + d\omega_3), g_3(a\omega_1 + b\omega_3, c\omega_1 + d\omega_3))$$
$$\sigma_3(z; g_2(a\omega_1 + b\omega_3, c\omega_1 + d\omega_3), g_3(a\omega_1 + b\omega_3, c\omega_1 + d\omega_3)) = \sigma_1(z; g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3))$$
$$\sigma_2(z; g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3))\ \sigma_3(z; g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)) \;/;\; \{a, b, c, d\} \in \mathbb{Z} \land a d - b c = \pm 1$$

Homogeneity

09.16.04.0008.01

$$\sigma_n(\lambda z; g_2(\lambda \omega_1, \lambda \omega_3), g_3(\lambda \omega_1, \lambda \omega_3)) = \sigma_n(z; g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)) \;/;\; n \in \{1, 2, 3\}$$

09.16.04.0009.01

$$\sigma_n(\lambda z; g_2(\lambda \omega_1, \lambda \omega_3), g_3(\lambda \omega_1, \lambda \omega_3)) = \sigma_n\left(\lambda z; \frac{g_2(\omega_1, \omega_3)}{\lambda^4}, \frac{g_3(\omega_1, \omega_3)}{\lambda^6}\right) \;/;\; n \in \{1, 2, 3\}$$

Poles and essential singularities

With respect to z

For fixed g_2, g_3, the functions $\sigma_j(z; g_2, g_3)$, $j \in \{1, 2, 3\}$ have simple poles at the points $z = 2 m \omega_1(g_2, g_3) + 2 n \omega_3(g_2, g_3)$, $\{m, n\} \in \mathbb{Z}$.

09.16.04.0010.01

$$\mathrm{Sing}_z(\sigma_j(z; g_2, g_3)) = \{\{\{2 m \omega_1 + 2 n \omega_3, 1\} \;/;\; \{m, n\} \in \mathbb{Z}\}\} \;/;\; j \in \{1, 2, 3\}$$

Branch points

With respect to z

For fixed g_2, g_3, the function $\sigma_j(z; g_2, g_3)$, $j \in \{1, 2, 3\}$ does not have branch points.

09.16.04.0011.01

$$\mathcal{BP}_z(\sigma_j(z; g_2, g_3)) = \{\} /; j \in \{1, 2, 3\}$$

Branch cuts

With respect to z

For fixed g_2, g_3, the function $\sigma_j(z; g_2, g_3)$, $j \in \{1, 2, 3\}$ does not have branch cuts.

09.16.04.0012.01

$$\mathcal{BC}_z(\sigma_j(z; g_2, g_3)) = \{\} /; j \in \{1, 2, 3\}$$

Series representations

q-series

q-series for logarithms

09.16.06.0001.01

$$\log(\sigma_1(z; g_2, g_3)) = \frac{\eta_1 z^2}{2\omega_1} + \log\left(\cos\left(\frac{\pi z}{2\omega_1}\right)\right) + 4\sum_{k=1}^{\infty} (-1)^k \frac{q^{2k}}{k(1-q^{2k})} \sin^2\left(\frac{k\pi z}{2\omega_1}\right)$$

09.16.06.0002.01

$$\log(\sigma_2(z; g_2, g_3)) = \frac{\eta_1 z^2}{2\omega_1} + 4\sum_{k=1}^{\infty} (-1)^k \frac{q^{k}}{k(1-q^{2k})} \sin^2\left(\frac{k\pi z}{2\omega_1}\right)$$

09.16.06.0003.01

$$\log(\sigma_3(z; g_2, g_3)) = \frac{\eta_1 z^2}{2\omega_1} + 4\sum_{k=1}^{\infty} \frac{q^{k}}{k(1-q^{2k})} \sin^2\left(\frac{k\pi z}{2\omega_1}\right)$$

Other series representations

09.16.06.0004.01

$$\sigma_i(z; g_2, g_3) = z \exp\left(-\sum_{j=2}^{\infty} \frac{z^{2j}}{2j} \sum_{\substack{m,n=-\infty \\ \{m,n\}\neq\{0,0\}}}^{\infty} \frac{1}{(2m\omega_1 + 2n\omega_3)^{2j}}\right)\left(-e_i + \frac{1}{z^2} + \sum_{j=1}^{\infty} (2j+1) z^{2j} \sum_{\substack{m,n=-\infty \\ \{m,n\}\neq\{0,0\}}}^{\infty} \frac{1}{(2m\omega_1 + 2n\omega_3)^{2j+2}}\right)^{1/2} /;$$
$$i \in \{1, 2, 3\}$$

Product representations

Infinite products involving trigonometric functions

09.16.08.0001.01

$$\sigma_i(z; g_2, g_3) = \exp\left(\frac{\eta_i z^2}{2\omega_i}\right)\cos\left(\frac{\pi z}{2\omega_i}\right)\prod_{n=1}^{\infty}\left(1 - \frac{\sin^2\left(\frac{\pi z}{2\omega_i}\right)}{\cos^2\left(\frac{n\pi\omega_j}{\omega_i}\right)}\right) /; \{i, j\} \in \{1, 2, 3\} \land i \neq j$$

09.16.08.0002.01

$$\sigma_i(z;\, g_2,\, g_3) \equiv \exp\left(\frac{\eta_j\, z^2}{2\,\omega_j}\right) \prod_{n=1}^{\infty} \left(1 - \frac{\sin^2\left(\frac{\pi\, z}{2\,\omega_j}\right)}{\cos^2\left(\frac{n\,\pi\,\omega_i}{\omega_j}\right)}\right) /;\ \{i,\, j\} \in \{1,\, 2,\, 3\} \land i \neq j$$

09.16.08.0003.01

$$\sigma_i(z;\, g_2,\, g_3) \equiv \exp\left(\frac{\eta_j\, z^2}{2\,\omega_j}\right) \prod_{n=1}^{\infty} \left(1 - \frac{\sin^2\left(\frac{\pi\, z}{2\,\omega_j}\right)}{\cos^2\left(\frac{2\,n-1}{2}\,\frac{\pi\,\omega_k}{\omega_j}\right)}\right) /;\ \{i,\, j,\, k\} \in \{1,\, 2,\, 3\} \land i \neq j \neq k$$

Infinite products involving exponentials

09.16.08.0004.01

$$\sigma_i(z;\, g_2,\, g_3) \equiv \exp\left(-\frac{e_i\, z^2}{2}\right) \prod_{\substack{m,\, n=-\infty \\ \{m,n\} \neq \{0,0\}}}^{\infty} \left(\left(1 - \frac{z}{2\,m\,\omega_1 + 2\,n\,\omega_2 - \omega_i}\right) \exp\left(\frac{z}{2\,m\,\omega_1 + 2\,n\,\omega_2 - \omega_i} + \frac{z^2}{2\,(2\,m\,\omega_1 + 2\,n\,\omega_2 - \omega_i)^2}\right)\right) /;$$

$$i \in \{1,\, 2,\, 3\}$$

Infinite products involving *q*, trigonometrics and exponentials

09.16.08.0005.01

$$\sigma_1(z;\, g_2,\, g_3) \equiv \cos\left(\frac{\pi\, z}{2\,\omega_1}\right) \exp\left(\frac{\eta_1\, z^2}{2\,\omega_1}\right) \left(\prod_{n=1}^{\infty} \frac{1 + q^{2\,n} \exp\left(-\frac{i\,\pi\, z}{\omega_1}\right)}{1 + q^{2\,n}}\right) \left(\prod_{n=1}^{\infty} \frac{1 + q^{2\,n} \exp\left(\frac{i\,\pi\, z}{\omega_1}\right)}{1 + q^{2\,n}}\right)$$

09.16.08.0006.01

$$\sigma_1(z;\, g_2,\, g_3) \equiv \cos\left(\frac{\pi\, z}{2\,\omega_1}\right) \exp\left(\frac{\eta_1\, z^2}{2\,\omega_1}\right) \prod_{n=1}^{\infty} \frac{1 + 2\, q^{2\,n} \cos\left(\frac{\pi\, z}{\omega_1}\right) + q^{4\,n}}{\left(1 + q^{2\,n}\right)^2}$$

09.16.08.0007.01

$$\sigma_2(z;\, g_2,\, g_3) \equiv \exp\left(\frac{\eta_1\, z^2}{2\,\omega_1}\right) \left(\prod_{n=1}^{\infty} \frac{1 + q^{2\,n-1} \exp\left(-\frac{i\,\pi\, z}{\omega_1}\right)}{1 + q^{2\,n-1}}\right) \left(\prod_{n=1}^{\infty} \frac{1 + q^{2\,n-1} \exp\left(\frac{i\,\pi\, z}{\omega_1}\right)}{1 + q^{2\,n-1}}\right)$$

09.16.08.0008.01

$$\sigma_2(z;\, g_2,\, g_3) \equiv \exp\left(\frac{\eta_1\, z^2}{2\,\omega_1}\right) \prod_{n=1}^{\infty} \frac{1 + 2\, q^{2\,n-1} \cos\left(\frac{\pi\, z}{\omega_1}\right) + q^{4\,n-2}}{\left(1 + q^{2\,n-1}\right)^2}$$

09.16.08.0009.01

$$\sigma_3(z;\, g_2,\, g_3) \equiv \exp\left(\frac{\eta_1\, z^2}{2\,\omega_1}\right) \left(\prod_{n=1}^{\infty} \frac{1 - q^{2\,n-1} \exp\left(-\frac{i\,\pi\, z}{\omega_1}\right)}{1 - q^{2\,n-1}}\right) \left(\prod_{n=1}^{\infty} \frac{1 - q^{2\,n-1} \exp\left(\frac{i\,\pi\, z}{\omega_1}\right)}{1 - q^{2\,n-1}}\right)$$

09.16.08.0010.01

$$\sigma_3(z;\, g_2,\, g_3) \equiv \exp\left(\frac{\eta_1\, z^2}{2\,\omega_1}\right) \prod_{n=1}^{\infty} \frac{1 - 2\, q^{2\,n-1} \cos\left(\frac{\pi\, z}{\omega_1}\right) + q^{4\,n-2}}{\left(1 - q^{2\,n-1}\right)^2}$$

q-products for half-period values

09.16.08.0011.01

$$\sigma_1(\omega_1;\, g_2,\, g_3) = 0$$

09.16.08.0012.01

$$\sigma_1(\omega_2;\, g_2,\, g_3) = -\frac{i\sqrt{i}}{2\sqrt[4]{q}} \exp\left(\frac{\eta_2\, \omega_2}{2}\right) \left(\prod_{n=1}^{\infty} \frac{1 - q^{2n-1}}{1 + q^{2n}}\right)^2$$

09.16.08.0013.01

$$\sigma_1(\omega_3;\, g_2,\, g_3) = \frac{1}{2\sqrt[4]{q}} \exp\left(\frac{\eta_3\, \omega_3}{2}\right) \left(\prod_{n=1}^{\infty} \frac{1 + q^{2n-1}}{1 + q^{2n}}\right)^2$$

09.16.08.0014.01

$$\sigma_2(\omega_1;\, g_2,\, g_3) = \exp\left(\frac{\eta_1\, \omega_1}{2}\right) \left(\prod_{n=1}^{\infty} \frac{1 - q^{2n-1}}{1 + q^{2n-1}}\right)^2$$

09.16.08.0015.01

$$\sigma_2(\omega_2;\, g_2,\, g_3) = 0$$

09.16.08.0016.01

$$\sigma_2(\omega_3;\, g_2,\, g_3) = 2\sqrt[4]{q}\, \exp\left(\frac{\eta_3\, \omega_3}{2}\right) \left(\prod_{n=1}^{\infty} \frac{1 + q^{2n}}{1 + q^{2n-1}}\right)^2$$

09.16.08.0017.01

$$\sigma_3(\omega_1;\, g_2,\, g_3) = \exp\left(\frac{\eta_1\, \omega_1}{2}\right) \left(\prod_{n=1}^{\infty} \frac{1 + q^{2n-1}}{1 - q^{2n-1}}\right)^2$$

09.16.08.0018.01

$$\sigma_3(\omega_2;\, g_2,\, g_3) = 2\sqrt{i}\, \sqrt[4]{q}\, \exp\left(\frac{\eta_2\, \omega_2}{2}\right) \left(\prod_{n=1}^{\infty} \frac{1 + q^{2n}}{1 - q^{2n-1}}\right)^2$$

09.16.08.0019.01

$$\sigma_3(\omega_3;\, g_2,\, g_3) = 0$$

Transformations

Addition formulas

Translation by half-periods

09.16.16.0001.01

$$\sigma_i(z \pm \omega_i;\, g_2,\, g_3) = \mp\, e^{\pm \eta_i z} \frac{\sigma_j(\omega_i;\, g_2,\, g_3)\, \sigma_k(\omega_i;\, g_2,\, g_3)\, \sigma(z;\, g_2,\, g_3)}{\sigma(\omega_i;\, g_2,\, g_3)} \,/;\, \{i,\, j,\, k\} \in \{1,\, 2,\, 3\} \land i \neq j \neq k$$

09.16.16.0002.01

$$\sigma_j(z \pm \omega_i;\, g_2,\, g_3) = e^{\pm \eta_i z}\, \sigma_j(\omega_i;\, g_2,\, g_3)\, \sigma_k(z;\, g_2,\, g_3) \,/;\, \{i,\, j,\, k\} \in \{1,\, 2,\, 3\} \land i \neq j \neq k$$

09.16.16.0003.01

$$\sigma_i(z_1 + z_2;\, g_2,\, g_3)\, \sigma_i(z_1 - z_2;\, g_2,\, g_3) = \sigma_i(z_1;\, g_2,\, g_3)^2\, \sigma_i(z_2;\, g_2,\, g_3)^2 - (e_i - e_j)(e_i - e_k)\, \sigma(z_1;\, g_2,\, g_3)^2\, \sigma(z_2;\, g_2,\, g_3)^2 \,/;\, \{i,\, j,\, k\} \in \{1,\, 2,\, 3\} \land i \neq j \neq k$$

09.16.16.0004.01

$$\sigma(z_1+z_2;g_2,g_3)\,\sigma(z_1-z_2;g_2,g_3)=\sigma(z_1;g_2,g_3)^2\,\sigma_i(z_2;g_2,g_3)^2-\sigma(z_2;g_2,g_3)^2\,\sigma_i(z_1;g_2,g_3)^2\;/;\;i\in\{1,2,3\}$$

09.16.16.0005.01

$$\sigma_i(z_1+z_2;g_2,g_3)\,\sigma_i(z_1-z_2;g_2,g_3)=\sigma_i(z_1;g_2,g_3)^2\,\sigma_j(z_2;g_2,g_3)^2-(e_i-e_j)\,\sigma(z_2;g_2,g_3)^2\,\sigma_k(z_1;g_2,g_3)^2\;/;\;\{i,j,k\}\in\{1,2,3\}\land i\neq j\neq k$$

09.16.16.0006.01

$$\sigma_i(z_1+z_2;g_2,g_3)\,\sigma(z_1-z_2;g_2,g_3)=\sigma(z_1;g_2,g_3)\,\sigma_i(z_1;g_2,g_3)\,\sigma_j(z_2;g_2,g_3)\,\sigma_k(z_2;g_2,g_3)-\sigma(z_2;g_2,g_3)\,\sigma_i(z_2;g_2,g_3)\,\sigma_j(z_1;g_2,g_3)\,\sigma_k(z_1;g_2,g_3)\;/;\;\{i,j,k\}\in\{1,2,3\}\land i\neq j\neq k$$

09.16.16.0007.01

$$\sigma_k(z_1+z_2;g_2,g_3)\,\sigma_j(z_1-z_2;g_2,g_3)=\sigma_k(z_1;g_2,g_3)\,\sigma_j(z_1;g_2,g_3)\,\sigma_k(z_2;g_2,g_3)\,\sigma_j(z_2;g_2,g_3)+(e_j-e_k)\,\sigma(z_1;g_2,g_3)\,\sigma_i(z_1;g_2,g_3)\,\sigma(z_2;g_2,g_3)\,\sigma_i(z_2;g_2,g_3)\;/;\;\{i,j,k\}\in\{1,2,3\}\land i\neq j\neq k$$

Related transformations

Halving half-period

09.16.16.0008.01

$$\sigma_1\left(z;g_2\left(\frac{\omega_1}{2},\omega_3\right),g_3\left(\frac{\omega_1}{2},\omega_3\right)\right)=\exp\left(\frac{e_1 z^2}{2}\right)\left(\sigma_1(z;g_2,g_3)^2-e^{\eta_1\omega_1}\frac{\sigma(z;g_2,g_3)^2}{\sigma(\omega_1;g_2,g_3)^2}\right)$$

09.16.16.0009.01

$$\sigma_2\left(z;g_2\left(\frac{\omega_1}{2},\omega_3\right),g_3\left(\frac{\omega_1}{2},\omega_3\right)\right)=\exp\left(\frac{e_1 z^2}{2}\right)\left(\sigma_1(z;g_2,g_3)^2+e^{\eta_1\omega_1}\frac{\sigma(z;g_2,g_3)^2}{\sigma(\omega_1;g_2,g_3)^2}\right)$$

09.16.16.0010.01

$$\sigma_3\left(z;g_2\left(\frac{\omega_1}{2},\omega_3\right),g_3\left(\frac{\omega_1}{2},\omega_3\right)\right)=\exp\left(\frac{e_1 z^2}{2}\right)\sigma_2(z;g_2,g_3)\,\sigma_3(z;g_2,g_3)$$

Third of half-period

09.16.16.0011.01

$$\sigma_i\left(z;g_2\left(\frac{\omega_1}{3},\omega_3\right),g_3\left(\frac{\omega_1}{3},\omega_3\right)\right)=\exp\left(z^2\,\wp\left(\frac{2\omega_1}{3};g_2,g_3\right)-2z\eta_1\right)\left(\sigma_i(z;g_2,g_3)\,\sigma_i\left(z+\frac{2\omega_1}{3};g_2,g_3\right)\sigma_i\left(z+\frac{4\omega_1}{3};g_2,g_3\right)\right)\Big/\left(\sigma_i\left(\frac{2\omega_1}{3};g_2,g_3\right)\sigma_i\left(\frac{4\omega_1}{3};g_2,g_3\right)\right)\;/;\;i\in\{1,2,3\}$$

General fractions of half-periods

09.16.16.0012.01

$$\sigma_1\left(z;g_2\left(\frac{\omega_1}{n},\omega_3\right),g_3\left(\frac{\omega_1}{n},\omega_3\right)\right)=\exp\left(\frac{(n-1)z\eta_1}{n}+\sum_{k=1}^{n-1}\left(\frac{z^2}{2}\wp\left(\frac{2k\omega_1}{n};g_2,g_3\right)-\frac{(2k-n+1)\eta_1 z}{n}\right)\right)\prod_{k=0}^{n-1}\frac{\sigma_1\left(z+\frac{(2k-n+1)\omega_1}{n};g_2,g_3\right)}{\sigma_1\left(\frac{(2k-n+1)\omega_1}{n};g_2,g_3\right)}\;/;\;n\in\mathbb{N}^+$$

09.16.16.0013.01

$$\sigma_2\left(z;\, g_2\left(\frac{\omega_1}{n},\, \omega_3\right),\, g_3\left(\frac{\omega_1}{n},\, \omega_3\right)\right) = \exp\left(\frac{(n-1)\, z\, \eta_1}{n} + \sum_{k=1}^{n-1}\left(\frac{z^2}{2}\, \wp\left(\frac{2\, k\, \omega_1}{n};\, g_2,\, g_3\right) - \frac{(2\, k - n + 1)\, \eta_1\, z}{n}\right)\right) \prod_{k=0}^{n-1} \frac{\sigma_2\left(z + \frac{(2\, k - n + 1)\, \omega_1}{n};\, g_2,\, g_3\right)}{\sigma_2\left(\frac{(2\, k - n + 1)\, \omega_1}{n};\, g_2,\, g_3\right)} \,/;\, n \in \mathbb{N}^+$$

09.16.16.0014.01

$$\sigma_3\left(z;\, g_2\left(\frac{\omega_1}{n},\, \omega_3\right),\, g_3\left(\frac{\omega_1}{n},\, \omega_3\right)\right) = \exp\left(\sum_{k=1}^{n-1}\left(\frac{z^2}{2}\, \wp\left(\frac{2\, k\, \omega_1}{n};\, g_2,\, g_3\right) - \frac{2\, k\, \eta_1\, z}{n}\right)\right) \prod_{k=0}^{n-1} \frac{\sigma_3\left(z + \frac{2\, k\, \omega_1}{n};\, g_2,\, g_3\right)}{\sigma_3\left(\frac{2\, k\, \omega_1}{n};\, g_2,\, g_3\right)} \,/;\, n \in \mathbb{N}^+$$

Identities

Functional identities

09.16.17.0001.01

$$\sigma_i(z;\, g_2,\, g_3)^2 - \sigma_j(z;\, g_2,\, g_3)^2 = (e_j - e_i)\, \sigma(z;\, g_2,\, g_3)^2 \,/;\, \{i,\, j\} \in \{1,\, 2,\, 3\} \land i \neq j$$

09.16.17.0002.01

$$(e_k - e_j)\, \sigma_i(z;\, g_2,\, g_3)^2 + (e_i - e_k)\, \sigma_j(z;\, g_2,\, g_3)^2 + (e_j - e_i)\, \sigma_k(z;\, g_2,\, g_3)^2 = 0 \,/;\, \{i,\, j\} \in \{1,\, 2,\, 3\} \land i \neq j$$

Differentiation

Low-order differentiation

With respect to *z*

09.16.20.0001.01

$$\frac{\partial \sigma_n(z,\, g_2,\, g_3)}{\partial z} = \sigma_n(z,\, g_2,\, g_3)\, (\zeta(z + \omega_n;\, g_2,\, g_3) - \eta_n)$$

09.16.20.0002.01

$$\frac{\partial^2 \sigma_n(z,\, g_2,\, g_3)}{\partial z^2} = \sigma_n(z,\, g_2,\, g_3)\left((\eta_n - \zeta(z + \omega_n;\, g_2,\, g_3))^2 - \wp(z + \omega_n;\, g_2,\, g_3)\right)$$

Symbolic differentiation

With respect to *z*

09.16.20.0003.01

$$\frac{\partial^n \sigma_m(z,\, g_2,\, g_3)}{\partial z^n} = \frac{(2\, \omega_1)^{1-n}\, \pi^{n-\frac{1}{2}}\, e^{-\eta_m z}}{\sigma_m(z,\, g_2,\, g_3)} \left(\prod_{k=1}^{\infty} \frac{1}{1 - q^{2k}}\right)^3 \sum_{j=0}^{n} {}_2\tilde{F}_2\left(\frac{1}{2},\, 1;\, \frac{1-j}{2},\, \frac{2-j}{2};\, \frac{(z + \omega_m)^2\, \zeta(\omega_1;\, g_2,\, g_3)}{2\, \omega_1}\right) \binom{n}{j} \left(\frac{4\, \omega_1}{\pi\, (z + \omega_m)}\right)^j$$
$$\sum_{k=0}^{\infty} (-1)^k\, q^{k(k+1)}\, (2\, k + 1)^{n-j} \sin\left(\frac{1}{2\, \omega_1}\, (\pi\, ((n - j)\, \omega_1 + (2\, k + 1)\, (z + \omega_m)))\right) - \eta_m\, \sigma_m(z,\, g_2,\, g_3) \,/;\, n \in \mathbb{N}^+$$

Representations through equivalent functions

With related functions

Involving other Weierstrass functions

09.16.27.0001.01

$$\sigma_i(z;\, g_2,\, g_3) \equal \sigma(z;\, g_2,\, g_3)\sqrt{\wp(z;\, g_2,\, g_3) - e_i} \;/;\; i \in \{1,\, 2,\, 3\}$$

09.16.27.0002.01

$$\frac{\sigma_i(z;\, g_2,\, g_3)^2}{\sigma(z;\, g_2,\, g_3)^2} \equal \wp(z;\, g_2,\, g_3) - e_i \;/;\; i \in \{1,\, 2,\, 3\}$$

Involving theta functions

09.16.27.0003.01

$$\sigma_1(z;\, g_2,\, g_3) \equal \frac{1}{2}\,\frac{1}{\sqrt[4]{q}}\left(\prod_{n=1}^{\infty}\frac{1}{\left(1 - q^{4n}\right)\left(1 + q^{2n}\right)}\right)\exp\!\left(\frac{\eta_1 z^2}{2\,\omega_1}\right)\vartheta_2\!\left(\frac{\pi z}{2\,\omega_1},\, q\right)$$

09.16.27.0004.01

$$\sigma_2(z;\, g_2,\, g_3) \equal \left(\prod_{n=1}^{\infty}\frac{1}{\left(1 - q^{2n}\right)\left(1 + q^{2n-1}\right)^2}\right)\exp\!\left(\frac{\eta_1 z^2}{2\,\omega_1}\right)\vartheta_3\!\left(\frac{\pi z}{2\,\omega_1},\, q\right)$$

09.16.27.0005.01

$$\sigma_3(z;\, g_2,\, g_3) \equal \left(\prod_{n=1}^{\infty}\frac{1}{\left(1 - q^{2n}\right)\left(1 - q^{2n-1}\right)^2}\right)\exp\!\left(\frac{\eta_1 z^2}{2\,\omega_1}\right)\vartheta_4\!\left(\frac{\pi z}{2\,\omega_1},\, q\right)$$

09.16.27.0006.01

$$\sigma_i(u;\, g_2,\, g_3) \equal \exp\!\left(\frac{\eta_1 z^2}{2\,\omega_1}\right)\frac{\vartheta_{i+1}\!\left(\frac{\pi z}{2\,\omega_1},\, q\right)}{\vartheta_{i+1}(0,\, q)} \;/;\; i \in \{1,\, 2,\, 3\}$$

Zeros

09.16.30.0001.01

$$\sigma_i\!\left((2\,m + 1)\,\omega_i + 2\,n\,\omega_j + 2\,r\,\omega_k;\, g_2,\, g_3\right) \equal 0 \;/;\; \{m,\, n,\, r\} \in \mathbb{Z} \land \{i,\, j,\, k\} \in \{1,\, 2,\, 3\} \land i \neq j \neq k$$

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